

Pendant-tree connectivity of line graphs*

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Abstract

The concept of pendant-tree connectivity, introduced by Hager in 1985, is a generalization of classical vertex-connectivity. In this paper, we study pendant-tree connectivity of line graphs.

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1 Introduction

A processor network is expressed as a graph, where a node is a processor and an edge is a communication link. Broadcasting is the process of sending a message from the source node to all other nodes in a network. It can be accomplished by message dissemination in such a way that each node repeatedly receives and forwards messages. Some of the nodes and/or links may be faulty. However, multiple copies of messages can be disseminated through disjoint paths. We say that the broadcasting succeeds if all the healthy nodes in the network finally obtain the correct message from the source node within a certain limit of time. A lot of attention has been devoted to fault-tolerant broadcasting in networks [11, 15, 17, 36]. In order to measure the ability of fault-tolerance, the above path structure connecting two nodes are generalized into some tree structures connecting more than two nodes, see [19, 21, 24]. To show these generalizations clearly, we must state from the connectivity in graph theory. We divide our introduction into the following four subsections to state the motivations and our results of this paper.

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1.1 Connectivity and k -connectivity

All graphs considered in this paper are undirected, finite and simple. We refer to the book [2] for graph theoretical notation and terminology not described here. For a graph G , let $V(G)$, $E(G)$, $e(G)$, $L(G)$ and $\delta(G)$ denote the set of vertices, the set of edges, the size, the line graph and the minimum degree of G , respectively. In the sequel, let $K_{s,t}$, K_n and P_n denote the complete bipartite graph of order $s+t$ with part sizes s and t , complete graph of order n , and path of order n , respectively. For any subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X , and $E[X]$ the edge set of $G[X]$. For two subsets X and Y of $V(G)$ we denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y . If $X = \{x\}$, we simply write $E_G[x, Y]$ for $E_G[\{x\}, Y]$.

Connectivity is one of the most basic concepts of graph-theoretic subjects, both in combinatorial sense and the algorithmic sense. It is well-known that the classical connectivity has two equivalent definitions. The *connectivity* of G , written $\kappa(G)$, is the minimum order of a vertex set $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. We call this definition the ‘cut’ version definition of connectivity. A well-known theorem of Whitney [38] provides an equivalent definition of connectivity, which can be called the ‘path’ version definition of connectivity. For any two distinct vertices x and y in G , the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y . Then $\kappa(G) = \min\{\kappa_G(x, y) \mid x, y \in V(G), x \neq y\}$ is defined to be the *connectivity* of G . For connectivity, Oellermann gave a survey paper on this subject; see [32].

Although there are many elegant and powerful results on connectivity in graph theory, the basic notation of classical connectivity may not be general enough to capture some computational settings. So people want to generalize this concept. For the ‘cut’ version definition of connectivity, we find the above minimum vertex set without regard the number of components of $G - S$. Two graphs with the same connectivity may have differing degrees of vulnerability in the sense that the deletion of a vertex cut-set of minimum cardinality from one graph may produce a graph with considerably more components than in the case of the other graph. For example, the star $K_{1,n}$ and the path P_{n+1} ($n \geq 3$) are both trees of order $n+1$ and therefore connectivity 1, but the deletion of a cut-vertex from $K_{1,n}$ produces a graph with n components while the deletion of a cut-vertex from P_{n+1} produces only two components. Chartrand et al. [6] generalized the ‘cut’ version definition of connectivity. For an integer k ($k \geq 2$) and a graph G of order n ($n \geq k$), the *k -connectivity* $\kappa'_k(G)$ is the smallest number of vertices whose removal from G of order n ($n \geq k$) produces a graph with at least k components or a graph with fewer than k vertices. Thus, for $k = 2$, $\kappa'_2(G) = \kappa(G)$. For more details about k -connectivity, we refer to [6, 18, 32, 33].

1.2 Generalized (edge-)connectivity

The generalized connectivity of a graph G , introduced by Hager [14], is a natural generalization of the ‘path’ version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an S -Steiner tree or a Steiner tree connecting S (or simply, an S -tree) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Note that when $|S| = 2$ an S -Steiner tree is just a path connecting the two vertices of S . Two S -Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa_G(S)$ is the maximum number of internally disjoint S -Steiner trees in G , that is, we search for the maximum cardinality of edge-disjoint trees which include S and are vertex disjoint with the exception of S . For an integer k with $2 \leq k \leq n$, *generalized k -connectivity* (or *k -tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ when S runs over all k -subsets of $V(G)$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G , that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G . By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when G is disconnected. Note that the generalized k -connectivity and k -connectivity of a graph are indeed different. Take for example, the graph H_1 obtained from a triangle with vertex set $\{v_1, v_2, v_3\}$ by adding three new vertices u_1, u_2, u_3 and joining v_i to u_i by an edge for $1 \leq i \leq 3$. Then $\kappa_3(H_1) = 1$ but $\kappa'_3(H_1) = 2$. There are many results on the generalized connectivity, see [7, 21, 22, 23, 24, 25, 26, 27, 28, 34].

As a natural counterpart of the generalized connectivity, we introduced the concept of generalized edge-connectivity in [28]. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint Steiner trees connecting S in G . For an integer k with $2 \leq k \leq n$, the *generalized k -edge-connectivity* $\lambda_k(G)$ of G is then defined as $\lambda_k(G) = \min\{\lambda(S) \mid S \subseteq V(G) \text{ and } |S| = k\}$. It is also clear that when $|S| = 2$, $\lambda_2(G)$ is just the standard edge-connectivity $\lambda(G)$ of G , that is, $\lambda_2(G) = \lambda(G)$, which is the reason why we address $\lambda_k(G)$ as the generalized edge-connectivity of G . Also set $\lambda_k(G) = 0$ when G is disconnected. Results on the generalized edge-connectivity can be found in [24, 27, 28].

1.3 Pendant-tree (edge-)connectivity

The concept of pendant-tree connectivity [14] was introduced by Hager in 1985, which is specialization of generalized connectivity (or k -tree-connectivity) but a generalization of classical connectivity. For an S -Steiner tree, if the degree of each vertex in S is equal to one, then this tree is called a *pendant S -Steiner tree*. Two pendant S -Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *pendant-tree local connectivity* $\tau_G(S)$ is the maximum number of internally disjoint pendant S -Steiner trees in G . For an integer k with $2 \leq k \leq n$,

pendant-tree k -connectivity is defined as $\tau_k(G) = \min\{\tau_G(S) \mid S \subseteq V(G), |S| = k\}$. When $k = 2$, $\tau_2(G) = \tau(G)$ is just the connectivity of a graph G . For more details on pendant-tree connectivity, we refer to [14, 29]. Clearly, we have

$$\begin{cases} \tau_k(G) = \kappa_k(G), & \text{for } k = 1, 2; \\ \tau_k(G) \leq \kappa_k(G), & \text{for } k \geq 3. \end{cases}$$

The relation between pendant-tree connectivity and generalized connectivity are shown in the following Table 2.

	Pendant tree-connectivity	Generalized connectivity
Vertex subset	$S \subseteq V(G) \ (S \geq 2)$	$S \subseteq V(G) \ (S \geq 2)$
Set of Steiner trees	$\begin{cases} \mathcal{T}_S = \{T_1, T_2, \dots, T_\ell\} \\ S \subseteq V(T_i), \\ d_{T_i}(v) = 1 \text{ for every } v \in S \\ E(T_i) \cap E(T_j) = \emptyset, \end{cases}$	$\begin{cases} \mathcal{T}_S = \{T_1, T_2, \dots, T_\ell\} \\ S \subseteq V(T_i), \\ E(T_i) \cap E(T_j) = \emptyset, \end{cases}$
Local parameter	$\tau(S) = \max \mathcal{T}_S $	$\kappa(S) = \max \mathcal{T}_S $
Global parameter	$\tau_k(G) = \min_{S \subseteq V(G), S =k} \tau(S)$	$\kappa_k(G) = \min_{S \subseteq V(G), S =k} \kappa(S)$

Table 2. Two kinds of tree-connectivities

The following two observations are easily seen.

Observation 1.1 *If G is a connected graph, then $\tau_k(G) \leq \mu_k(G) \leq \delta(G)$.*

Observation 1.2 *If H is a spanning subgraph of G , then $\tau_k(H) \leq \tau_k(G)$.*

In [14], Hager derived the following results.

Lemma 1.1 [14] *Let G be a graph. If $\tau_k(G) \geq \ell$, then $\delta(G) \geq k + \ell - 1$.*

Lemma 1.2 [14] *Let G be a graph. If $\tau_k(G) \geq \ell$, then $\kappa(G) \geq k + \ell - 2$.*

Lemma 1.3 [14] *Let k, n be two integers with $3 \leq k \leq n$, and let K_n be a complete graph of order n . Then*

$$\tau_k(K_n) = n - k.$$

As a natural counterpart of the pendant-tree k -connectivity, we introduced the concept of pendant-tree k -edge-connectivity. For $S \subseteq V(G)$ and $|S| \geq 2$, the *pendant-tree local edge-connectivity* $\mu(S)$ is the maximum number of edge-disjoint pendant S -Steiner trees in G . For an integer k with $2 \leq k \leq n$, the *pendant-tree k -edge-connectivity* $\mu_k(G)$ of G is then defined as $\mu_k(G) = \min\{\mu(S) \mid S \subseteq V(G) \text{ and } |S| = k\}$. It is also clear that when $|S| = 2$, $\mu_2(G)$ is just the standard edge-connectivity $\lambda(G)$ of G , that is, $\mu_2(G) = \lambda(G)$.

1.4 Application background and our results

In addition to being a natural combinatorial measure, both the pendant-tree connectivity and the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G , then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI (see [12, 13, 35]) and computer communication networks (see [10]). Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k -connectivity can serve for measuring the capability of a network G to connect any k vertices in G .

Chartrand and Stewart [8] investigated the relation between the connectivity and edge-connectivity of a graph and its line graph.

Theorem 1.1 [8] *If G is a connected graph, then*

- (1) $\kappa(L(G)) \geq \lambda(G)$ if $\lambda(G) \geq 2$.
- (2) $\lambda(L(G)) \geq 2\lambda(G) - 2$.
- (3) $\kappa(L(L(G))) \geq 2\kappa(G) - 2$.

In Section 2, we investigate the relation between the pendant-tree 3-connectivity and pendant-tree 3-edge-connectivity of a graph and its line graph.

In their book, Capobianco and Molluzzo [4], using $K_{1,n}$ as their example, note that the difference between the connectivity of a graph and its line graph can be arbitrarily large. They then proposed an open problem: Whether for any two integers p, q ($1 < p < q$), there exists a graph G such that $\kappa(G) = p$ and $\kappa(L(G)) = q$. In [1], Bauer and Tindell gave a positive answer of this problem, that is, for every pair of integers p, q ($1 < p < q$) there is a graph of connectivity p whose line graph has connectivity q .

Note that the difference between the pendant-tree k -connectivity of a graph G and its line graph $L(G)$ can be also arbitrarily large. Let n, k be two integers with $2 \leq k \leq n$, and $G = K_{1,n}$. Then $L(G) = K_n$, $\tau_k(G) = 0$ and $\tau_k(L(G)) = n - k$. In fact, we can consider a similar problem: Whether for any two integers p, q , $1 < p < q$, there exists a graph G such that $\tau_k(G) = p$ and $\tau_k(L(G)) = q$. It seems to be not easy to solve this problem for a general k . In this paper, we focus our attention on the case $k = 3$, and give a positive answer of this problem.

2 Preliminary

In [14], Hager showed that τ_k is monotonically decreasing for k .

Lemma 2.1 [14] *Let G be a graph, and let k be an integer with $k \geq 2$. Then*

$$\tau_k(G) \geq \tau_{k+1}(G).$$

By a result in [14], Mao and Lai obtained the following bounds of $\tau_k(G)$.

Lemma 2.2 [29] *Let G be a graph of order n , and let k be an integer with $3 \leq k \leq n$. Then*

$$\frac{1}{k+1} \log_2 \kappa(G) \leq \tau_k(G) \leq \kappa(G).$$

Moreover, the bounds are sharp.

Proposition 2.1 [30] *Let k, n be two integers with $3 \leq k \leq n$, and let G be a graph. Then*

$$0 \leq \tau_k(G) \leq n - k.$$

Moreover, the bounds are sharp.

For $k = n, n-1, n-2$, the following corollaries are immediate.

Lemma 2.3 [30] *Let G be a graph of order n . Then $\tau_n(G) = 0$ if and only if G is a graph of order n .*

Lemma 2.4 [30] *Let G be a connected graph of order n . Then*

- (1) $\tau_{n-1}(G) = 1$ if and only if G is a complete graph of order n .
- (2) $\tau_{n-1}(G) = 0$ if and only if G is not a complete graph of order n .

Lemma 2.5 [30] *Let G be a connected graph of order n . Then*

- (1) $\tau_{n-2}(G) = 2$ if and only if G is a complete graph of order n .
- (2) $\tau_{n-2}(G) = 1$ if and only if $G = K_n \setminus M$ and $1 \leq |M| \leq 2$, where M is a matching of K_n for $n \geq 7$.
- (3) $\tau_{n-2}(G) = 0$ if and only if G is one of the other graphs.

The following results for line graphs can be found in [37].

Lemma 2.6 [5, 16] *For $n \neq 8$, $L(K_n)$ is the only $(2n-4)$ -regular simple graph of order $\binom{n}{2}$ in which nonadjacent vertices have four common neighbors and adjacent vertices have $n-2$ common neighbors.*

Lemma 2.7 ([37], p-283) *Let G be a k -edge-connected simple graph. Then $L(G)$ is k -connected and $(2k-2)$ -edge-connected.*

Let S be a set of k vertices of a connected graph G , and let \mathcal{T} be a set of edge-disjoint pendant S -Steiner trees. Then the following observation is immediate.

Observation 2.1 *Let k, n be two integers with $3 \leq k \leq n$. Let G be a graph of order n , and let $S \subseteq V(G)$ with $|S| = k$. For each $T \in \mathcal{T}$,*

$$|E(T) \cap E_G[S, \bar{S}]| \geq k,$$

where $\bar{S} = V(G) - S$.

By the above result, we can easily derive an upper bound for pendant-tree k -edge-connectivity.

Observation 2.2 *For any graph G with order at least k ,*

$$\mu_k(G) \leq \min_{S \subseteq V(G), |S|=k} \left\lfloor \frac{1}{k} |E_G[S, \bar{S}]| \right\rfloor,$$

where $S \subseteq V(G)$ with $|S| = k$, and $\bar{S} = V(G) - S$. Moreover, the bound is sharp.

Proposition 2.2 *Let k, n be two integers with $3 \leq k \leq n$, and let K_n be a complete graph of order n . Then*

$$\mu_k(K_n) = n - k.$$

Proof. From Observation 2.2, we have $\mu_k(K_n) \leq \min_{S \subseteq V(K_n), |S|=k} \left\lfloor \frac{1}{k} |E_{K_n}[S, \bar{S}]| \right\rfloor = \frac{1}{k}(n - k)k = n - k$. From Lemma 1.3 and Observation 1.1, we have $\mu_k(K_n) \geq \tau_k(K_n) = n - k$. So $\mu_k(K_n) = n - k$, as desired. \blacksquare

Note that each graph is a spanning subgraph of a complete graph. So the following result is immediate.

Proposition 2.3 *Let k, n be two integers with $3 \leq k \leq n$, and let G be a graph. Then*

$$0 \leq \mu_k(G) \leq n - k.$$

Graphs with $\mu_k(G) = n - k$ are characterized as follows.

Proposition 2.4 *Let G be a graph of order n . Then $\mu_k(G) = n - k$ if and only if G is a complete graph.*

Proof. If G is a complete graph, then it follows from Proposition 2.2 that $\mu_k(G) \geq \tau_k(G) \geq n - k$. From Proposition 2.3, we have $\mu_k(G) = n - k$. Conversely, we suppose $\mu_k(G) = n - k$. We claim that G is a complete graph. Assume, to the contrary, that G is not a complete graph. From Observation 2.2, we have

$$\mu_k(G) \leq \min_{S \subseteq V(G), |S|=n-2} \left\lfloor \frac{1}{k} |E_{K_n}[S, \bar{S}]| \right\rfloor \leq \frac{1}{k}[k(n - k) - 1] = n - k - \frac{1}{k},$$

and hence $\mu_k(G) \leq n - k - 1$, a contradiction. So G is a complete graph of order n , as desired.

The following two corollaries are immediate from Propositions 2.3 and 2.4.

Corollary 2.1 *Let G be a graph of order n . Then $\mu_n(G) = 0$ if and only if G is a graph of order n .*

Corollary 2.2 *Let G be a graph of order n . Then*

- (1) $\mu_{n-1}(G) = 1$ if and only if G is a complete graph.
- (2) $\mu_{n-1}(G) = 0$ if and only if G is not a complete graph.

From the above corollary, we can get the relation between $\mu_{n-1}(G)$ and $\lambda(G)$.

Proposition 2.5 *Let G be a graph of order n . Then*

$$\mu_{n-1}(G) = \left\lfloor \frac{\lambda(G)}{n-1} \right\rfloor.$$

Proof. If G is a complete graph, then it follows from Corollary 2.2 that $\lambda(G) = n - 1$ and $\mu_{n-1}(G) = 1$, and hence $\mu_{n-1}(G) = 1 = \lfloor \frac{\lambda(G)}{n-1} \rfloor$. If G is not a complete graph, then it follows from Corollary 2.2 that $0 \leq \lambda(G) \leq n - 2$ and $\mu_{n-1}(G) = 0$, and hence $\mu_{n-1}(G) = 0 = \lfloor \frac{\lambda(G)}{n-1} \rfloor$, as desired. \blacksquare

From Proposition 2.3, we have $0 \leq \mu_{n-2}(G) \leq 2$. In the following, graphs with $\mu_{n-2}(G) = \ell$ ($0 \leq \ell \leq 2$) are characterized.

Lemma 2.8 *Let G be a connected graph of order n . Then $\mu_{n-2}(G) = 1$ if and only if $\bar{G} = K_{1,r} \cup (n - r - 1)K_1$, where $1 \leq r \leq n - 2$.*

Proof. Suppose that $\bar{G} = K_{1,r} \cup (n - r - 1)K_1$ and $1 \leq r \leq n - 2$. From Proposition 2.4, we have $\mu_{n-2}(G) \leq 1$. Let u be the center of $K_{1,r}$. For any $S \subseteq V(G)$ and $|S| = n - 2$, we have $u \in S$ or $u \in \bar{S}$, where $\bar{S} = V(G) - S$. If $u \in \bar{S}$, then we let $\bar{S} = \{u, v\}$. Since $\bar{G} = K_{1,r} \cup (n - r - 1)K_1$, it follows that the tree induced by the edges in $E_G[v, S]$ is a pendant S -Steiner tree, and hence $\mu(S) \geq 1$. From now on, we suppose $u \in S$. Set $\bar{S} = \{v, w\}$. Suppose that $u \in S$. Suppose $uv \notin E(G)$ and $uw \notin E(G)$. Since $\bar{G} = K_{1,r} \cup (n - r - 1)K_1$ where $1 \leq r \leq n - 2$, it follows that there exists a vertex in S , say x , such that $ux \in E(G)$. Then the tree induced by the edges in $\{ux\} \cup E_G[w, S] - \{uw\}$ is a pendant S -Steiner tree, and hence $\mu(S) \geq 1$. Suppose $uv \notin E(G)$ or $uw \notin E(G)$. Without loss of generality, let $uv \notin E(G)$. Then $uw \in E(G)$, and the tree induced by the edges in $E_G[w, S]$ is a pendant S -Steiner tree, and hence $\mu(S) \geq 1$. From the argument, we conclude that $\mu(S) \geq 1$ for any $S \subseteq V(G)$ and $|S| = n - 2$. From the arbitrariness of S , we have $\mu_{n-2}(G) \geq 1$, and hence $\mu_{n-2}(G) = 1$.

Conversely, we suppose $\mu_{n-2}(G) = 1$. Then we have the following claim.

Claim 1. \bar{G} does not contain a P_4 as its subgraph.

Proof of Claim 1: Assume, to the contrary, that \bar{G} contains a $P_4 = u_1u_2u_3u_4$ as its subgraph. Choose $S = V(G) - \{u_2u_3\}$. Since $u_iu_{i+1} \notin E(G)$ ($1 \leq i \leq 3$), it follows that there is no S -Steiner tree in G , a contradiction.

From Claim 1, \bar{G} does not contain cycles of order at least 4 as its subgraph. Furthermore, we have the following claim.

Claim 2. \bar{G} does not contain cycles.

Proof of Claim 2: Assume, to the contrary, that \bar{G} contains a cycle. From Claim 1, this cycle is a triangle, say $C_3 = uvw$. Choose $S = V(G) - \{u, w\}$. Since $uv, uw, vw \notin E(G)$, it follows that there is no pendant S -Steiner tree in G , a contradiction.

From Claim 2, \bar{G} is a tree. From Claim 1, \bar{G} is a star. Set $\bar{G} = K_{1,r}$. Since $\mu_{n-2}(G) = 1$, it follows that G is connected, and hence $1 \leq r \leq n-2$, as desired. ■

From Lemma 2.8 and Proposition 2.4, the following result is easily seen.

Proposition 2.6 *Let G be a connected graph of order n . Then*

- (1) $\mu_{n-2}(G) = 2$ if and only if G is a complete graph of order n .
- (2) $\mu_{n-2}(G) = 1$ if and only if $\bar{G} = K_{1,r} \cup (n-r-1)K_1$ where $1 \leq r \leq n-2$.
- (3) $\mu_{n-2}(G) = 0$ if and only if $e(\bar{G}) \geq 1$ and $\bar{G} \neq K_{1,r} \cup (n-r-1)K_1$ and $1 \leq r \leq n-2$.

From Proposition 2.6, we can set up the relation between $\mu_{n-2}(G)$ and $\lambda(G)$.

Proposition 2.7 *Let G be a graph of order n . Then*

$$\mu_{n-2}(G) = \begin{cases} \left\lceil \frac{\lambda(G)}{n-2} \right\rceil & \text{if } \lambda(G) = n-1, \\ & \text{or } \bar{G} = K_{1,r} \cup (n-r-1)K_1 \text{ where } 1 \leq r \leq n-2, \\ \left\lfloor \frac{\lambda(G)}{n-2} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof. If $\lambda(G) = n-1$, then G is a complete graph, and hence $\mu_{n-2}(G) = 2 = \lceil \frac{\lambda(G)}{n-2} \rceil$. If $\bar{G} = K_{1,r} \cup (n-r-1)K_1$ where $1 \leq r \leq n-2$, then it follows from Proposition 2.6 that $\mu_{n-2}(G) = 1 = \lceil \frac{\lambda(G)}{n-2} \rceil$. For other cases, from Proposition 2.6, we have $\mu_{n-2}(G) = 0 = \lfloor \frac{\lambda(G)}{n-2} \rfloor$. ■

Corollary 2.3 *Let G be a graph of order n . Then*

$$\left\lfloor \frac{\lambda(G)}{n-2} \right\rfloor \leq \mu_{n-2}(G) \leq \left\lceil \frac{\lambda(G)}{n-2} \right\rceil.$$

In [29], Mao and Lai got the following results.

Lemma 2.9 [29] *Let G be a connected graph with n vertices. Then*

$$\frac{1}{12}\kappa(G) - \frac{1}{2} \leq \tau_3(G) \leq \frac{2}{3}\kappa(G).$$

Moreover, the lower bound is sharp.

Lemma 2.10 [29] *Let G be a graph of order n . Then*

$$\tau_{n-2}(G) = \begin{cases} \left\lceil \frac{\kappa(G)}{n-2} \right\rceil & \text{if } \kappa(G) = n-1, \\ & \text{or } \kappa(G) = n-2 \text{ and } \bar{G} = iK_2 \cup (n-2i)K_1 \ (i=1,2), \\ \left\lfloor \frac{\kappa(G)}{n-2} \right\rfloor & \text{otherwise.} \end{cases}$$

Lemma 2.11 *Let G be a graph of order n . Then*

$$\tau_{n-1}(G) = \left\lfloor \frac{\kappa(G)}{n-1} \right\rfloor.$$

3 Pendant-tree connectivity of line graphs

For pendant-tree k -connectivity, we have the following:

Proposition 3.1 *If G is a connected graph, then*

- (1) $\frac{1}{k+1} \log_2 \mu_k(G) \leq \tau_k(L(G)).$
- (2) $\mu_k(L(G)) \geq \frac{1}{k+1} \log_2 \mu_k(G).$
- (3) $\tau_k(L(L(G))) \geq \frac{1}{k+1} (1 + \log_2(\tau_k(G) - 1)).$

Proof. For (1), from Lemma 2.2 and Theorem 1.1, we have

$$\tau_k(L(G)) \geq \frac{1}{k+1} \log_2 \kappa(L(G)) \geq \frac{1}{k+1} \log_2 \lambda(G) \geq \frac{1}{k+1} \log_2 \mu_k(G).$$

For (2), from (1) of this proposition, we have

$$\mu_k(L(G)) \geq \tau_k(L(G)) \geq \frac{1}{k+1} \log_2 \mu_k(G).$$

For (3), from (1) of this proposition and Lemma 2.2, we have

$$\tau_k(L(L(G))) \geq \frac{1}{k+1} \log_2 \kappa(L(L(G))) \geq \frac{1}{k+1} \log_2(2\kappa(G)-2) \geq \frac{1}{k+1} (1 + \log_2(\tau_k(G) - 1)).$$

As we have seen, the above bounds are relatively rough. In the following, we try to improve the bounds for $k = n, n-1, n-2, 3$.

Proposition 3.2 *Let G be a connected graph of order n . Then*

- (1) $\mu_n(G) \leq \tau_n(L(G))$.
- (2) $\mu_n(L(G)) \geq \mu_n(G)$.
- (3) $\tau_n(L(L(G))) \geq \tau_n(G)$.

Proof. For (1), we set $|E(G)| = m$. If $m \geq n$, then it follows from Corollary 2.2 and Lemma 2.1 that $\mu_n(G) = 0 = \tau_m(L(G)) \leq \tau_n(L(G))$. If $m = n - 1$, then $\mu_n(G) = 0 \leq \tau_n(L(G))$. For (2), from (1), we have $\mu_n(L(G)) \geq \tau_n(L(G)) \geq \mu_n(G)$. For (3), from (1) and (2), we have $\tau_n(L(L(G))) \geq \mu_n(L(G)) \geq \mu_n(G) \geq \tau_n(G)$. \blacksquare

Proposition 3.3 *Let G be a connected graph of order n ($n \geq 3$). If G is 2-edge-connected, then*

- (1) $\left\lceil \frac{2\mu_{n-1}(G)}{n} \right\rceil - 1 \leq \tau_{n-1}(L(G))$;
- (2) $\mu_{n-1}(L(G)) \geq \left\lfloor \frac{4\mu_{n-1}(G)-4}{(n+1)(n-2)} \right\rfloor$;
- (3) $\tau_{n-1}(L(L(G))) \geq \left\lfloor \frac{4\tau_{n-1}(G)-4}{(n+1)(n-2)} \right\rfloor$.

Proof. For (1), we set $|E(G)| = m$. Since G is 2-connected, it follows that $m \geq n$. From Lemma 2.1 and Theorem 1.1 that

$$\begin{aligned} \mu_{n-1}(G) &= \left\lfloor \frac{\lambda(G)}{n-1} \right\rfloor \leq \left\lfloor \frac{\kappa(L(G))}{n-1} \right\rfloor \leq \frac{m}{n-1} \frac{\kappa(L(G))}{m} \\ &\leq \frac{m}{n-1} \left(\left\lfloor \frac{\kappa(L(G))}{m} \right\rfloor + 1 \right) = \frac{m}{n-1} (\tau_{m-1}(G) + 1) \\ &\leq \frac{n}{2} (\tau_{n-1}(G) + 1), \end{aligned}$$

and hence

$$\tau_{n-1}(L(G)) \geq \left\lceil \frac{2\mu_{n-1}(G)}{n} \right\rceil - 1.$$

For (2), from Lemma 2.1, Theorem 1.1 and Proposition 2.4, we have

$$\mu_{n-1}(L(G)) \geq \mu_m(L(G)) = \left\lfloor \frac{\lambda(L(G))}{m-1} \right\rfloor \geq \left\lfloor \frac{2\lambda(G)-2}{\binom{n}{2}-1} \right\rfloor \geq \left\lfloor \frac{4\mu_{n-1}(G)-4}{(n+1)(n-2)} \right\rfloor.$$

For (3), from Lemmas 2.1, 2.11 and Theorem 1.1, we have

$$\tau_{n-1}(L(L(G))) \geq \tau_m(L(L(G))) = \left\lfloor \frac{\kappa(L(L(G)))}{m-1} \right\rfloor \geq \left\lfloor \frac{2\kappa(G)-2}{\binom{n}{2}-1} \right\rfloor \geq \left\lfloor \frac{4\tau_{n-1}(G)-4}{(n+1)(n-2)} \right\rfloor.$$

Proposition 3.4 *Let G be a connected graph of order n . If G is 2-edge-connected, then*

- (1) $\left\lfloor \frac{2\mu_{n-2}(G)}{n^2-n-4} \right\rfloor \leq \tau_{n-2}(L(G)).$
- (2) $\mu_{n-2}(L(G)) \geq \left\lfloor \frac{4\mu_{n-2}(G)-4}{n^2-n-4} \right\rfloor.$
- (3) $\tau_{n-2}(L(L(G))) \geq \left\lfloor \frac{4\tau_{n-2}(G)-4}{n^2-n-4} \right\rfloor.$

Proof. (1) From Theorem 1.1, Proposition 2.7 and Lemma 2.11, we have

$$\tau_{n-2}(L(G)) \geq \tau_{m-2}(L(G)) \geq \left\lfloor \frac{\kappa(L(G))}{m-2} \right\rfloor \geq \left\lfloor \frac{\lambda(G)}{m-2} \right\rfloor \geq \left\lfloor \frac{2\mu_{n-2}(G)}{n^2-n-4} \right\rfloor.$$

For (2), from (1) of this proposition and Proposition 2.4, we have

$$\mu_{n-1}(L(G)) \geq \mu_m(L(G)) \geq \left\lfloor \frac{\lambda(L(G))}{m-2} \right\rfloor \geq \left\lfloor \frac{2\lambda(G)-2}{\binom{n}{2}-2} \right\rfloor \geq \left\lfloor \frac{4\mu_{n-2}(G)-4}{n^2-n-4} \right\rfloor,$$

where m is the size of G .

For (3), from (1) of this proposition and Corollary 2.2, we have

$$\tau_{n-2}(L(L(G))) \geq \tau_m(L(L(G))) = \left\lfloor \frac{\kappa(L(L(G)))}{m-2} \right\rfloor \geq \left\lfloor \frac{2\kappa(G)-2}{\binom{n}{2}-2} \right\rfloor \geq \left\lfloor \frac{4\tau_{n-1}(G)-4}{n^2-n-4} \right\rfloor.$$

For pendant-tree 3-connectivity, we have the following:

Theorem 3.1 *Let G be a connected graph. Then*

- (1) $\mu_3(G) \leq \tau_3(L(G)).$
- (2) $\mu_3(L(G)) \geq \frac{1}{12}\mu_3(G) - \frac{1}{2}.$
- (3) $\tau_3(L(L(G))) \geq \frac{1}{4}\tau_3(G) - \frac{2}{3}.$

Proof. For (1), let e_1, e_2, e_3 be three arbitrary distinct vertices of the line graph of G such that $\mu_3(G) = \ell$ with $\ell \geq 1$. Let $e_1 = v_1v'_1$, $e_2 = v_2v'_2$ and $e_3 = v_3v'_3$ be those edges of G corresponding to the vertices e_1, e_2, e_3 in $L(G)$, respectively.

Consider three distinct vertices of the six end-vertices of e_1, e_2, e_3 . Without loss of generality, let $S = \{v_1, v_2, v_3\}$ be three distinct vertices. Since $\mu_3(G) = \ell$, there exist ℓ edge-disjoint pendant S -Steiner trees in G , say T_1, T_2, \dots, T_ℓ . We define a minimal S -Steiner tree T as an S -Steiner tree whose subtree obtained by deleting any edge of T does not connect S .

By choosing any two edge-disjoint minimal S -Steiner trees T_i and T_j ($1 \leq i, j \leq \ell$) in G , we will show that the trees T'_i and T'_j corresponding to T_i and T_j in $L(G)$ are internally disjoint pendant S -Steiner trees. It is easy to see that $T_i \cup T_j$ has three possible types, as

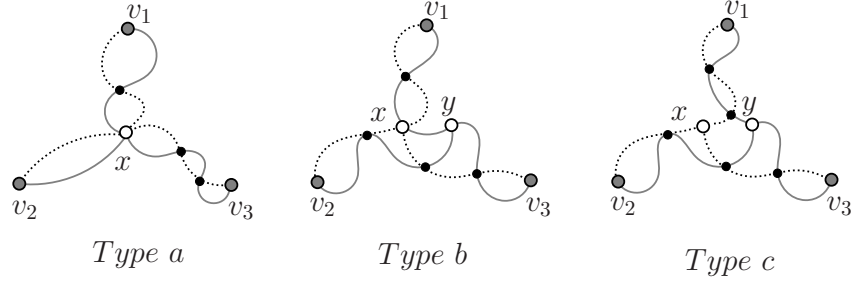


Figure 3.1 Three possible types of $T_i \cup T_j$.

shown in Figure 3.1. Since T_i and T_j are edge-disjoint in G , we can find internally disjoint pendant Steiner trees T'_i and T'_j connecting e_1, e_2, e_3 in $L(G)$. We give an example of Type a ; see Figure 3.2. So $\tau_3(L(G)) \geq \ell$, as desired.

For (2), from Lemma 2.9 and Theorem 1.1, we have

$$\mu_3(L(G)) \geq \tau_3(L(G)) \geq \frac{1}{12}\kappa(L(G)) - \frac{1}{2} \geq \frac{1}{12}\lambda(G) - \frac{1}{2} \geq \frac{1}{12}\mu_3(G) - \frac{1}{2}.$$

For (3), from Lemma 2.9 and Theorem 1.1, we have

$$\begin{aligned} \tau_3(L(L(G))) &\geq \frac{1}{12}\kappa(L(L(G))) - \frac{1}{2} \geq \frac{1}{12}(2\kappa(G) - 2) - \frac{1}{2} \\ &= \frac{1}{6}\kappa(G) - \frac{2}{3} \geq \frac{1}{4}\tau_3(G) - \frac{2}{3}. \end{aligned}$$

4 Graphs with prescribed pendant-tree connectivity and pendant-tree edge-connectivity

In [14], Hager obtained the following result.

Lemma 4.1 [14] *Let $K_{a,b}$ be a complete bipartite graph with $a + b$ vertices. Then*

$$\tau_k(K_{a,b}) = \max\{\min\{a - k + 1, b - k + 1\}, 0\}.$$

The following corollary is immediate from the above lemma.

Corollary 4.1 *Let a, b be two integers with $2 \leq a \leq b$, and $K_{a,b}$ denote a complete bipartite graph with a bipartition of sizes a and b , respectively. Then*

$$\tau_3(K_{a,b}) = a - 2.$$

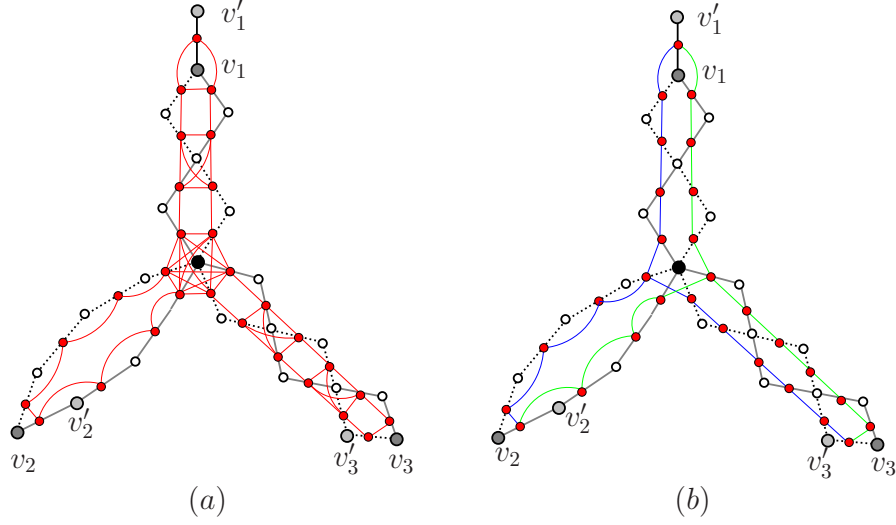


Figure 3.2 (a): An example for T_i and T_j connecting S and their line graphs; (b): An example for T'_i and T'_j corresponding to T_i and T_j .

Mao and Lai obtained the following result.

Lemma 4.2 [29] *Let G be a connected graph with minimum degree δ . Then*

$$\mu_k(G) \leq \delta(G) - k + 1.$$

In this section, we consider the above problem for the case $p \geq 2q$. Let us put our attention on the complete bipartite graph $G = K_{p+2, q-p+2}$. Since $q \geq 2p$, it follows that $q - p + 2 \geq p + 2$. From Corollary 4.1, $\kappa_3(G) = \kappa_3(K_{p+2, q-p+2}) = p$. Now we turn to consider the line graph of the complete bipartite graph $G = K_{p+2, q-p+2}$.

Recall that the Cartesian product (also called the *square product*) of two graphs G and H , written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G \square H \cong H \square G$. The following lemma is easily seen.

Lemma 4.3 [37] *For a complete bipartite graph $K_{r,s}$, $L(K_{r,s}) = K_r \square K_s$.*

From the above lemma, $L(G) = L(K_{p+2, q-p+2}) = K_{p+2} \square K_{q-p+2}$. In order to obtain the exact value of $\tau_3(L(G)) = \tau_3(K_{p+2} \square K_{q-p+2})$, we consider to determine the exact value of the Cartesian product of two complete graphs.

Before proving Lemma 4.4, we define some notation. Let G and H be two graphs with $V(G) = \{u_1, u_2, \dots, u_r\}$ and $V(H) = \{v_1, v_2, \dots, v_s\}$, respectively. Then $V(G \square H) =$

$\{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G \square H$ induced by the vertex set $\{(u_i, v) \mid 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G \square H$ induced by the vertex set $\{(u, v_j) \mid 1 \leq j \leq m\}$.

Lemma 4.4 *Let r, s be two integers. Then*

$$\tau_3(K_r \square K_s) = r + s - 4.$$

Proof. Let $G = K_r$ and $H = K_s$. Set $V(G) = \{u_1, u_2, \dots, u_r\}$ and $V(H) = \{v_1, v_2, \dots, v_s\}$. From Lemma 1.3, we have $\tau_3(G) = \tau_3(K_r) = r - 3$ and $\tau_3(H) = \tau_3(K_s) = s - 3$. On one hand, we assume $x, y \in V(G(v_1))$. Then $d_{G \square H}(x) = d_{G \square H}(y) = \delta(G \square H) = r + s - 2$. Since $xy \in E(G \square H)$, it follows that $\tau_3(G \square H) \leq \delta(G \square H) - 2 \leq r + s - 4$.

On the other hand, we will show that $\tau_3(G \square H) \geq r + s - 4$. We need to show that for any $S = \{x, y, z\} \subseteq V(G \square H)$, there exist $r + s - 4$ internally disjoint pendant S -Steiner trees. We complete our proof by the following three cases.

Case 1. x, y, z belongs to the same $V(H(u_i))$ ($1 \leq i \leq r$).

Without loss of generality, we assume $x, y, z \in V(H(u_1))$. Since $\kappa_3(H) = s - 3$, there exist $s - 3$ internally disjoint pendant S -Steiner trees T_1, T_2, \dots, T_{s-3} in $H(u_1)$. Let x_j, y_j, z_j be the vertices corresponding to x, y, z in $H(u_j)$ ($2 \leq j \leq r$). Then the trees T'_j induced by the edges in $\{xx_j, yy_j, zz_j, x_jy_j, y_jz_j\}$ ($2 \leq j \leq r$) are $r - 1$ internally disjoint pendant S -Steiner trees. These trees together with the trees T_1, T_2, \dots, T_{s-3} are $r + s - 4$ internally disjoint pendant S -Steiner trees.

Case 2. Only two vertices of $\{x, y, z\}$ belong to some copy $H(u_i)$.

We may assume $x, y \in V(H(u_1))$, $z \in V(H(u_2))$. Let x', y' be the vertices corresponding to x, y in $H(u_2)$, and let z' be the vertex corresponding to z in $H(u_1)$. Without loss of generality, let $V(H(u_1)) = \{v_1, v_2, \dots, v_s\}$ and $V(H(u_2)) = \{v'_1, v'_2, \dots, v'_s\}$.

Suppose $z' \notin \{x, y\}$. Without loss of generality, let $\{x, y, z'\} = \{v_1, v_2, v_3\}$ in $H(u_1)$. Then the tree T_1 induced by the edges in $\{zz', xz', yz'\}$ and T_{i-2} induced by the edges in $\{xv_i, yv_i, v_iv'_i, v'_iz\}$ ($4 \leq i \leq s$) are $s - 2$ internally disjoint pendant S -Steiner trees. Let x_j, y_j, z_j be the vertices corresponding to x, y, z' in $H(u_j)$ ($3 \leq j \leq r$). Then the trees T'_j induced by the edges in $\{xx_j, yy_j, zz_j, x_jz_j, y_jz_j\}$ ($3 \leq j \leq s$) are $r - 2$ internally disjoint pendant S -Steiner trees. These trees together with the trees T_1, T_2, \dots, T_{r-2} are $r + s - 4$ internally disjoint pendant S -Steiner trees.

Suppose $z' \in \{x, y\}$. Without loss of generality, assume $z' = y$. Without loss of generality, let $\{x, y\} = \{v_1, v_2\}$ in $H(u_1)$. Then the trees T_{i-2} induced by the edges in $\{xv_i, yv_i, v_iv'_i, v'_iz\}$ ($3 \leq i \leq s$) are $s - 2$ internally disjoint pendant S -Steiner trees. Let x_j, y_j be the vertices corresponding to x, y in $H(u_j)$ ($3 \leq j \leq r$). Then the trees T'_j induced by the edges in $\{xx_j, yy_j, zy_j, x_jy_j\}$ ($3 \leq j \leq r$) are $r - 2$ internally disjoint pendant S -Steiner trees. These trees together with the trees T_1, T_2, \dots, T_{r-2} are $r + s - 4$ internally disjoint pendant S -Steiner trees.

Case 3. x, y, z are contained in distinct $H(u_i)$ s.

We may assume that $x \in V(H(u_1))$, $y \in V(H(u_2))$, $z \in V(H(u_3))$. Let y', z' be the vertices corresponding to y, z in $H(u_1)$, x', z'' be the vertices corresponding to x, z in $H(u_2)$ and x'', y'' be the vertices corresponding to x, y in $H(u_3)$. Without loss of generality, let $V(H(u_1)) = \{v_1, v_2, \dots, v_s\}$, $V(H(u_2)) = \{v'_1, v'_2, \dots, v'_s\}$, $V(H(u_3)) = \{v''_1, v''_2, \dots, v''_s\}$.

Suppose that x, y', z' are distinct vertices in $H(u_1)$. Without loss of generality, let $\{x, y', z'\} = \{v_1, v_2, v_3\}$ in $H(u_1)$, $\{x', y, z''\} = \{v'_1, v'_2, v'_3\}$ in $H(u_2)$ and $\{x'', y'', z\} = \{v''_1, v''_2, v''_3\}$ in $H(u_3)$. Then the tree T_1 induced by the edges in $\{xx', x'y, x'x'', x''z\}$, the tree T_2 induced by the edges in $\{xz', z'z'', yz'', z''z\}$, the tree T_3 induced by the edges in $\{xy', yy', yy'', y''z\}$ and the trees T_i induced by the edges in $\{xu_i, yu'_i, u'_i z, u_i u'_i, u'_i u''_i\}$ ($4 \leq i \leq s$) are s internally disjoint pendant S -Steiner trees. Let x_j, y_j, z_j be the vertices corresponding to x, y, z in $H(u_j)$ ($4 \leq j \leq r$). The the trees T'_j induced by the edges in $\{xx_j, yy_j, zz_j, x_j y_j, y_j z_j\}$ ($4 \leq j \leq r$) are $r - 3$ internally disjoint pendant S -Steiner trees. These trees together with the trees T_1, T_2, \dots, T_r are $r + s - 3$ internally disjoint pendant S -Steiner trees.

Suppose that two of x, y', z' are the same vertex in $H(u_1)$. Without loss of generality, let $y' = z'$, $\{x, y'\} = \{v_1, v_2\}$ in $H(u_1)$, $\{x', y\} = \{v'_1, v'_2\}$ in $H(u_2)$ and $\{x'', z\} = \{v''_1, v''_2\}$ in $H(u_3)$. Then the tree T_1 induced by the edges in $\{xx', x'y, x'x'', x''z\}$ and the trees T_{i-1} induced by the edges in $\{xv_i, yv'_i, zv''_i, v_i v'_i, v'_i v''_i\}$ ($2 \leq i \leq s$) are $s - 1$ internally disjoint pendant S -Steiner trees. Let x_j, y_j be the vertices corresponding to x, y in $H(u_j)$ ($4 \leq j \leq r$). The the trees T'_j induced by the edges in $\{xx_j, yy_j, zy_j, x_j y_j\}$ ($4 \leq j \leq r$) are $r - 3$ internally disjoint pendant S -Steiner trees. These trees together with the trees T_1, T_2, \dots, T_{s-1} are $r + s - 4$ internally disjoint pendant S -Steiner trees.

Suppose that x, y', z' are the same vertex in $H(u_1)$. Without loss of generality, let $x = y' = z'$, $x = v_1$ in $H(u_1)$, $y = v'_1$ in $H(u_2)$ and $z = v''_1$ in $H(u_3)$. Then the tree T_{i-1} induced by the edges in $\{xv_i, yv'_i, zv''_i, v_i v'_i, v'_i v''_i\}$ ($2 \leq i \leq s$) are $s - 1$ internally disjoint pendant S -Steiner trees. Let x_j be the vertices corresponding to x in $H(u_j)$ ($4 \leq j \leq r$). The the trees T'_j induced by the edges in $\{xx_j, yx_j, zx_j\}$ ($4 \leq j \leq r$) are $r - 3$ internally disjoint pendant S -Steiner trees. These trees together with the trees T_1, T_2, \dots, T_{s-1} are $r + s - 4$ internally disjoint pendant S -Steiner trees.

From the above argument, we conclude that for any $S = \{x, y, z\} \subseteq V(G \square H)$, there exist $r + s - 3$ internally disjoint pendant S -Steiner trees, and hence $\tau(S) \geq r + s - 4$. From the arbitrariness of S , we have $\tau_k(G) = r + s - 4$, as desired. \blacksquare

From Lemma 4.4, if $G = K_{p+2, q-p+2}$, then $\tau_3(L(G)) = \tau_3(K_{p+2} \square K_{q-p+2}) = (p+2) + (q-p+2) - 4 = q$. So the following result holds.

Theorem 4.1 *For any two integers p, q with $p \geq 2q$, there exists a graph G such that $\tau_3(G) = p$ and $\tau_3(L(G)) = q$.*

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